Inner Product Spaces

Consider $V = (V, g)$ a finite dimensional inner product space, where g is a symmetric, positive definite $(0, 2)$-tensor. Often, we will write $g(v, w) = \langle v, w\rangle$. Fix a basis $\{v_i\}$ of V, and declare

$$g_{ij} := \langle v_i, v_j\rangle.$$

Bundling g_{ij} into a matrix g, we observe g is invertible by non-degeneracy of the inner product, and g is positive-define and symmetric. If V, W are individually inner product spaces, so is their tensor product $V \otimes W$ by linearly extending the following formula on simple tensors

$$\langle v \otimes w, v' \otimes w'\rangle = \langle v, v'\rangle \cdot \langle w, w'\rangle.$$

In this way, we get inner products on all tensor powers of V. We extend the inner product to one on V^* by respecting the evaluation pairing

$$\begin{aligned} \mathrm{eval} : V^* \otimes V &\to \mathbb{R} \\ \omega \otimes v &\mapsto \omega(v). \end{aligned}$$

Note that the choice of a basis on V induces the dual basis $\{v^i\}$ on V^*. We define the inner product on the dual basis $\langle v^i, v^j\rangle$ to be ij-th entry of the inverse metric to be

$$g^{ij} = \langle v^i, v^j\rangle := (g^{-1})_{ij}$$

and extend linearly. Since $g^* = g$ and $\{\lambda_i\}$ the eigenvalues of g are positive, then $(g^{-1})^* = (g^*)^{-1} = g^{-1}$ and $\{\lambda_i^{-1}\}$ the eigenvalues of g^{-1} are also positive. In other words, g^{-1} defines an inner product on V^*. In the basis $\{v_i\}$, the metric g^{ij} satisfies

$$\begin{aligned} \sum_j g^{ij} g_{jk} &= \sum_j \langle v^i, v^j\rangle \langle v_j, v_k\rangle \\ &= \sum_j \mathrm{eval}^{\otimes 2}(v^i \otimes v_j, v^j \otimes v_k) \\ &= \sum_j \mathrm{eval}(v^i \otimes v_j) \cdot \mathrm{eval}(v^j \otimes v_k) \\ &= \sum_j v^i(v_j) \cdot v^j(v_k) \\ &= \sum_j \delta^i_j \delta^j_k \\ &= \delta^i_k. \end{aligned}$$

In this fashion, we induce inner products on all tensor powers of V and V^*. In particular, we get an inner product on endomorphisms of V, namely on the set of $(1,1)$-tensors $V^* \otimes V$. Recall for a basis $\{v_i\}$ of V, the linear map $T : V \to V$ such that

$$Tv_i = \sum_j v^j(Tv_i)v_j = T_i{}^j v_j,$$

corresponds to the following element of $V^* \otimes V$,

$$T = \sum_{i,j} T_i{}^j v^i \otimes v_j.$$

In particular, we calculate

$$\begin{aligned}
\langle T, T\rangle &= \langle \sum_{i,j} T_i{}^j v^i \otimes v_j, \sum_{k,l} T_k{}^l v^k \otimes v_l\rangle \\
&= \sum_{i,j,k,l} T_i{}^j T_k{}^l \langle v^i \otimes v_j, v^k \otimes v_l\rangle \\
&= \sum_{i,j,k,l} T_i{}^j T_k{}^l g^{ik} g_{jl} \\
&= \sum_{i,j,k,l} (g^{ik} T_k{}^l)(T_i{}^j g_{jl}) \\
&= \sum_{i,l} T^{il} T_{il}.
\end{aligned}$$

Alternatively, one may compute as

$$\begin{aligned}
\langle T, T\rangle &= \sum_{i,j,k,l} T_i{}^j (T_k{}^l g^{ik}) g_{jl} \\
&= \sum_{i,j,l} T_i{}^j T^{il} g_{jl} \\
&= \sum_{i,j} T_i{}^j T^i{}_j.
\end{aligned}$$

Following the previous computation, one sees the inner product on $(1,1)$-tensors is the trace inner product

$$\langle S, T\rangle = \mathrm{tr}(ST^*).$$

This inner product is stable under type change. For example, we define the $(2,0)$-version of T as

$$T = \sum_{i,j,k} g^{ik} T_k{}^j v_i \otimes v_j,$$

and we compute

$$\begin{aligned}
\langle T, T\rangle &= \langle \sum_{i,j,k} g^{ik} T_k{}^j v_i \otimes v_j, \sum_{l,m,n} g^{ml} T_l{}^n v_m \otimes v_n\rangle \\
&= \sum_{i,j,k,l,m,n} g^{ik} T_k{}^j g^{ml} T_l{}^n \langle v_i \otimes v_j, v_m \otimes v_n\rangle \\
&= \sum_{i,j,k,l,m,n} g^{ik} T_k{}^j g^{ml} T_l{}^n g_{im} g_{jn} \\
&= \sum_{i,j,k,l,m,n} (g^{ik} T_k{}^j)(g_{jn} T_l{}^n)(g_{im} g^{ml}) \\
&= \sum_{i,j,l} T^{ij} T_{lj} \delta_i^l \\
&= \sum_{i,j} T^{ij} T_{ij}.
\end{aligned}$$

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