Consequences of the First Variation Formula

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ABSTRACT. These are notes on 1.3 from Colding and Minocozzi's book "A Course on Minimal Surfaces" [1]. I originally prepared these notes for the Geometry and Analysis Student Seminar (GASS) at UConn.

1. The First Variation Formula

We briefly review section 1 of [1] and recall the first variation formula.

Given a real-valued C^2-function u defined on an appropriate region $\Omega \subseteq \mathbb{R}^2$, recall its **graph** $\Gamma(u)$ is defined to be the set of points $(x, y, u(x, y)) \in \Omega \times \mathbb{R}$. There is a functional $A\Gamma : C^2(\Omega) \to \mathbb{R}$ given by

$$A\Gamma(u) = \int_\Omega \sqrt{1 + \|\nabla u\|^2}$$

called the **area functional**. Using techniques in the calculus of variations, one finds the cirtical points of $A\Gamma$ are precisely the functions u satisfying the so-called **minimal surface equation**

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) = 0. \tag{1.1}$$

In fact, it can be shown that the critical points are even area-minimizing. Consequently, if u is a critical point of $A\Gamma$ (i.e. u satisfies the minimal surface equation (1.1)), then the surface $\Gamma(u)$ is said to be a **minimal surface.**

The procedure described can be generalized to submanifolds Σ immersed in a Riemannian manifold (M, g) with a covariant derivative ∇. Here Σ is said to be a minimal submanifold if and only if its mean curvature vanishes identically (one recovers (1.1) when $M = \mathbb{R}^n$). This condition is equivalent to

$$-\int_\Sigma \langle X, H\rangle = \int_\Sigma \text{div}_\Sigma(X) = 0 \tag{1.2}$$

for all compactly supported vector fields X along Σ vanishing on the boundary $\partial\Sigma$. Equation (1.2) is called the **first variation formula.**

The purpose of these notes is to discuss three consequences of (1.2), in particular the harmonocity of coordinates, the monotonicity formula, and the mean value inequality.

2. Coordinate Functions are Harmonic

Recall that a function u on $\Sigma \subseteq \mathbb{R}^n$ is said to be **harmonic on** Σ if $\Delta_\Sigma u = 0$, where $\Delta_\Sigma u = \mathrm{div}_\Sigma(\nabla_\Sigma u)$ is the Laplacian of u.

Proposition 2.1. *Let $\mathbb{R}^n$ be coordinatized by (x_i). Then $\Sigma^k \subseteq \mathbb{R}^n$ is a minimal submanifold if and only if the restrictions of the x_i to Σ are harmonic.*

Proof. This is a direct consequence of the first variation formula, so we need a compactly supported vector field along Σ that encodes information of the coordinates. Let $\eta \in \mathscr{C}^\infty_{\mathrm{cpt}}(\Sigma)$ be a compactly supported smooth function such that $\eta|_{\partial\Sigma} = 0$. Consider the vector field ηe_i, where $e_i = \nabla_\Sigma x_i$ is the i-th coordinate vector field along Σ. By the Leibniz rule,

$$\mathrm{div}_\Sigma(\eta e_i) = \langle \nabla_\Sigma \eta, e_i\rangle + \eta\ \mathrm{div}_\Sigma(e_i).$$

But e_i is a constant vector field, so the above becomes $\mathrm{div}_\Sigma(\eta e_i) = \langle \nabla_\Sigma \eta, e_i\rangle$. The first variation formula implies that

$$-\int_\Sigma \langle \eta e_i, H\rangle = \int_\Sigma \mathrm{div}_\Sigma(\eta e_i) = \int_\Sigma \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i\rangle. \tag{2.1}$$

On the other hand, the Divergence theorem gives

$$0 = \int_{\partial\Sigma} \eta \langle \nabla_\Sigma x_i, N\rangle = \int_\Sigma \mathrm{div}_\Sigma(\eta \nabla_\Sigma x_i) = \int_\Sigma \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i\rangle + \int_\Sigma \eta\ \Delta_\Sigma x_i \tag{2.2}$$

where N is the unique outward-pointing unit vector on $\partial\Sigma$. Combining (2.2) with (2.1), we have the following condition:

$$\int_\Sigma \langle \eta e_i, H\rangle = \int_\Sigma \eta\ \Delta_\Sigma x_i.$$

Since the statement "Σ is minimal" is equivalent to $H = 0$, this just says that Σ is minimal if and only if the restriction of x_i to Σ is harmonic.

□

This result can be used to uncover many properties of minimal submanifolds, especially regarding its shape. First recall that the (Euclidean, closed) half-spaces $H(a, e) \subseteq \mathbb{R}^n$ are the sets given by

$$H(a, e) = \{x \in \mathbb{R}^n : \langle x, e\rangle \le a\}$$

for each $a \in \mathbb{R}$ and $e \in \mathbb{S}^{n-1}$. Geometrically, the half-space $H(a, (1, 0))$ in $\mathbb{R}^2$ is the half plane left of the line $x = a$, and one views in general $H(a, e)$ as rotations of this plane along a circle.

Definition 2.2. Let $K \subseteq \mathbb{R}^n$ be a compact subset. Define the **convex hull of** K, denoted by $\mathrm{Conv}(K)$, to be the (convex) set

$$\mathrm{Conv}(K) = \bigcap_{H(a,e)\supseteq K} H(a, e).$$

The convex hull of a compact set K is the smallest convex set containing K. Compact minimal submanifolds satisfy a so-called **convex hull property:**

Corollary 2.3 (due to R. Osserman [2]). *Let $\Sigma^k \subseteq \mathbb{R}^n$ be a compact minimal submanifold. Then Σ is contained entirely within the convex hull of its boundary $\partial\Sigma$.*

Proof. Fix an $e \in \mathbb{S}^{n-1}$. Define a function $p_e : \Sigma \to \mathbb{R}$ given by $p_e(x) = \langle x, e \rangle$. By proposition (2.1), one has $\Delta_\Sigma p_e = 0$. The maximum principle implies the existence of a number $a \in \mathbb{R}$ such that $p_e(x) \leq a$ for all $x \in \Sigma$. Equivalently, if $x \in \Sigma$, then $\langle x, e \rangle \leq a$, hence $\Sigma \subseteq \text{Conv}(\partial\Sigma)$.

□

This gives a bit of an idea on how minimal submanifolds are shaped. Furthermore, the harmonicity of coordinate functions also have something to say about the homology of Σ:

Corollary 2.4. *Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold. Then for each $1 \leq i \leq k$, there are homomorphisms $F_i : H_{k-1}(\Sigma) \to \mathbb{R}$ given by*

$$F_i([\gamma]) = \int_\gamma \langle n_\gamma, e_i \rangle$$

where n_γ is the oriented conormal vector to γ (i.e., n_γ is normal to γ while tangent to Σ).

Proof. Linearity is clear, so we just show that the F_i are well-defined as maps. Given homologous $(k-1)$-chains γ_1 and γ_2, there is a k-chain Ω with $\partial\Omega = \gamma_1 - \gamma_2$ by definition. Geometrically, one views Ω as a region in $\mathbb{R}^n$ contained within Σ bounded by γ_1 and γ_2. By proposition 2.1 and an application of the Divergence theorem,

$$0 = \int_\Omega \Delta_\Sigma x_i = \int_{\partial\Omega} \langle e_i, N \rangle = \int_{\gamma_1} \langle e_i, n_{\gamma_1} \rangle - \int_{\gamma_2} \langle e_i, n_{\gamma_2} \rangle.$$

Therefore $F_i([\gamma_1]) = F_i([\gamma_2])$, proving the claim.

□

One can, roughly, view $F_i([\gamma])$ as a measure of the total flux in the i-th direction through the part of Σ determined by γ.

3. The Monotonicity Formula

We move to another corollary to the first variation formula, called the monotonicity formula. The formula asserts that the volume and density (which will later be defined) of minimal submanifolds obey a monotony law. First, we recall without proof the coarea formula, which is roughly a generalized Fubini's theorem:

Theorem 3.1. *Let h be a real-valued proper Lipschitz function on a manifold Σ. Then for any locally integrable function $f : \Sigma \to \mathbb{R}$, the following holds for all $t \in \mathbb{R}$:*

$$\int_{\{h \leq t\}} f \|\nabla_\Sigma h\| = \int_{-\infty}^t \int_{h=\tau} f \; d\tau \tag{3.1}$$

Finally, it is helpful to make the following observation. Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold; in particular, recall that $\text{div}_\Sigma Y^N = -\langle Y^N, H \rangle = 0$ for any vector field Y. Together with the fact that $\nabla_{e_j} x_i = \langle e_j, e_i \rangle = \delta_{ij}$, one has:

$$\Delta_\Sigma \|x\|^2 = 2\text{div}_\Sigma (x_1, \ldots, x_n)^\top = 2\text{div}_\Sigma (x_1, \ldots, x_n) = 2k. \tag{3.2}$$

We now state and prove the monotonicity formula.

Proposition 3.2 (Monotonicity Formula). *Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold. Fix a point $x_0 \in \mathbb{R}^n$. Finally, put $B_r = B^n_r(x_0)$ for the n-ball about x_0 of radius $r > 0$ and $V(A)$ for the volume of the subset $A \subseteq \mathbb{R}^n$. Then for all $0 < s < t$, the following holds:*

$$\frac{V(B_t \cap \Sigma)}{t^k} - \frac{V(B_s \cap \Sigma)}{s^k} = \int_{(B_t - B_s)\cap\Sigma} \frac{\|(x-x_0)^N\|^2}{\|x-x_0\|^{k+2}}. \tag{3.3}$$

Proof. There is a distance function $d : \Sigma \to \mathbb{R}$ given by $d(x) = \|x - x_0\|$. Note that

$$V(B_s \cap \Sigma) = V\{x \in \Sigma : \|x - x_0\| < s\} = V\{d \le s\}.$$

The advantage of working with d is that an application of the coarea formula is suddenly more feasible. Differentiating one of the terms on the LHS of the monotonicity formula gives

$$\frac{d}{ds}\big(s^{-k}V\{d \le s\})\big) = -ks^{-k-1}V\{d \le s\} + s^{-k}\frac{d}{ds}V\{d \le s\}. \tag{3.4}$$

We move to express each term in (3.4) in terms of $x - x_0$. By (3.2) and Stoke's theorem,

$$2kV\{d \le s\} = \int_{\{d\le s\}} \Delta_\Sigma d^2 = \int_{\{d=s\}} \big\langle \nabla_\Sigma d^2, N\big\rangle = 2\int_{\{d=s\}} \|(x-x_0)^\top\|. \tag{3.5}$$

On the other hand, the coarea formula gives

$$V\{d \le s\} = \int_{\{d\le s\}} 1 = \int_{\{d\le s\}} \frac{1}{\|\nabla_\Sigma d\|} \cdot \|\nabla_\Sigma d\| = \int_0^s \int_{d=\tau} \frac{1}{\|\nabla_\Sigma d\|}\, d\tau.$$

Now since $\|\nabla_\Sigma d\| = \|\nabla_\Sigma(\|x - x_0\|)\| = \|(x - x_0)^\top\|/\|x - x_0\|$, this becomes

$$V\{d \le s\} = \int_0^s \int_{\{d=\tau\}} \frac{\|x-x_0\|}{\|(x-x_0)^\top\|}\, d\tau. \tag{3.6}$$

Combining (3.5) and (3.6) into (3.4) and using the fact that the integration region is along $d(x) = \|x - x_0\| = s$, one computes

$$\begin{aligned}
\frac{d}{ds}\big(s^{-k}V\{d \le s\}\big) &= -s^{-k-1}\int_{\{d=s\}} \|(x-x_0)^\top\| + s^{-k}\cdot\frac{s}{s}\int_{\{d=s\}} \frac{\|x-x_0\|}{\|(x-x_0)^\top\|} \\
&= s^{-k-1}\int_{\{d=s\}} \left(\frac{\|x-x_0\|^2}{\|(x-x_0)^\top\|} - \|(x-x_0^\top)\|\right) \\
&= s^{-k-1}\int_{\{d=s\}} \frac{\|(x-x_0)^N\|^2}{\|(x-x_0)^\top\|} \\
&= \int_{\{d=s\}} \frac{\|(x-x_0)^N\|^2}{\|(x-x_0)^\top\|\cdot\|x-x_0\|^{k+1}}
\end{aligned}$$

It is helpful here to multiply the integrand by $\|\nabla_\Sigma d\|/\|\nabla_\Sigma d\|$:

$$\begin{aligned}
\frac{d}{ds}\big(s^{-k}V\{d \le s\}\big) &= \int_{\{d=s\}} \frac{\|(x-x_0)^N\|^2}{\|(x-x_0)^\top\|\cdot\|x-x_0\|^{k+1}}\cdot\frac{\|(x-x_0)^\top\|}{\|x-x_0\|}\cdot\frac{1}{\|\nabla_\Sigma d\|} \\
&= \int_{\{d=s\}} \frac{\|(x-x_0)^N\|^2}{\|x-x_0\|^{k+2}}\cdot\frac{1}{\|\nabla_\Sigma d\|}
\end{aligned}$$

Finally, integrate along s and apply the coarea formula once more:

$$s^{-k}V\{d \le s\} = \int_0^s \int_{\{d=\tau\}} \frac{\|(x-x_0)^N\|^2}{\|x-x_0\|^{k+2}} \cdot \frac{1}{\|\nabla_\Sigma d\|}\, d\tau = \int_{\{d\le s\}} \frac{\|(x-x_0)^N\|^2}{\|x-x_0\|^{k+2}}.$$

The case for the term $t^{-k}V\{d \le t\}$ is identical. Subtracting these two results in the monotonicity formula.

□

Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold and fix an $x_0 \in \mathbb{R}^n$. Define a function $\Theta_{x_0} : \mathbb{R}_{>0} \to \mathbb{R}$ by the formula

$$\Theta_{x_0}(s) = \frac{V(B_s^n(x_0) \cap \Sigma)}{V(B_s^k(x_0))}.$$

Note that the denominator is the volume of the k-ball, while in the numerator one takes the n-ball and intersects with the k-dimensional submanifold Σ.

Definition 3.3. Let x_0 be a point in a minimal submanifold. The **density at $\mathbf{x_0}$** is the quantity $\Theta_{x_0} = \lim_{s\to 0} \Theta_{x_0}(s)$.

Intuitively, one expects the density at any point x_0 of a minimal submanifold Σ to be at least 1, with equality holding if and only if Σ is dilation invariant about x_0. Indeed, we have the following due to the monotonicity formula:

Corollary 3.4. *Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold with a fixed point $x_0 \in \mathbb{R}^n$. Then the following hold:*

(1) *The function $\Theta_{x_0}(s)$ is monotone nondecreasing.*
(2) *The function $\Theta_{x_0}(s)$ is constant if and only if Σ is dilation invariant about x_0.*
(3) *If $x_0 \in \Sigma$, then $\Theta_{x_0}(s) \ge 1$.*

Proof. These are immediate consequences of the monotonicity formula. Note also that $\|(x-x_0)^N\| = 0$ for $x \in \Sigma$ precisely when Σ is conical about x_0.

□

One can extend definition 3.3 to include points in $\mathbb{R}^n$, thereby allowing discussion of the density at points outside of Σ. This gives a real-valued function on $\mathbb{R}^n$ defined by $x \mapsto \Theta_x$. As a consequence of the monotonicity formula, minimal submanifolds have the property that the density does not rise "too abruptly". More precisely:

Corollary 3.5. *Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold. Then the function $\Theta : \mathbb{R}^n \to \mathbb{R}$ given by $\Theta(x) = \Theta_x$ is upper semicontinuous.*

Proof. Fix an $x \in \mathbb{R}^n$ and let $(x_j)_{j=0}^\infty$ be any sequence of points converging to x. By corollary 3.4, given a $\varepsilon > 0$, one finds an $s > 0$ such that $\Theta_x \ge \Theta_x(2s) - \varepsilon$. Now choose a $\delta > 0$ such that $\delta < s$ and

$$\Theta_x \ge \left(1 + \frac{\delta}{s}\right)^k \Theta_x(2s) - 2\varepsilon.$$

For a point x_j with $\|x - x_j\| < \delta$, the following holds by corollary 3.4:

$$\Theta_{x_j} \le \Theta_{x_j}(s) \le \frac{V(B_{s+\delta}^n(x) \cap \Sigma)}{V(B^k(x))} = \left(1 + \frac{\delta}{s}\right)^k \Theta_x(s+\delta) \le \Theta_x + 2\varepsilon$$

In particular $\limsup_{x_j \to x} \Theta_{x_j} \le \Theta_x$, so Θ is upper semicontinuous. □

4. The Mean Value Inequality

In the previous section, we proved that minimal submanifolds have monotonicity in volume. It turns out that it is possible to generalize the monotonicity formula to include *weighted* volume, resulting in the mean value inequality. We conclude with the statement and proof of this.

Given a subset $A \subseteq \mathbb{R}^n$ and a weight function w defined on A, denote by $V_w(A)$ the volume of A with weight w. More precisely we have

$$V_w(A) = \int_A w.$$

PROPOSITION 4.1 (Mean Value Inequality). *Let $\Sigma^k \subseteq \mathbb{R}^n$ be a minimal submanifold. Given a function w on Σ, the following holds for all $0 < s < t$:*

$$\begin{aligned}\frac{V_w(B_t \cap \Sigma)}{t^k} - \frac{V_w(B_s \cap \Sigma)}{s^k} &= \int_{(B_t - B_s)\cap\Sigma} w \frac{\|x^N\|^2}{\|x\|^{k+2}} \\ &\quad + \frac{1}{2}\int_s^t \tau^{-k-1}\int_{B_\tau \cap \Sigma} (\tau^2 - \|x\|^2)\Delta_\Sigma w \; d\tau\end{aligned}$$

Observe that the mean value inequality reduces to the monotonicity formula when $w = 1$.

PROOF. The proof is similar to the proof of the monotonicity formula. Therefore, we outline most of this proof and write a few of the different aspects in detail. Consider once again the expression

$$\frac{d}{ds}\big(s^{-k}V_w(B_s \cap \Sigma)\big) = -ks^{-k-1}V_w(B_s\cap\Sigma) + s^{-k}\frac{d}{ds}V_w(B_s \cap \Sigma).$$

As before, we attempt to rewrite the RHS in terms of x. First it is helpful to write down the following:

$$\begin{aligned}\operatorname{div}_\Sigma\big(w\nabla_\Sigma\|x\|^2 - \|x\|^2\nabla_\Sigma w\big) &= \big\langle \nabla_\Sigma w, \Delta_\Sigma \|x\|^2\big\rangle + w\Delta_\Sigma\|x\|^2 \\ &\quad - \big\langle \nabla_\Sigma\|x\|^2, \nabla_\Sigma w\big\rangle - \|x\|^2\Delta_\Sigma w \\ \\ &= w\Delta_\Sigma\|x\|^2 - \|x\|^2\Delta_\Sigma w.\end{aligned}$$

Using this along with the Divergence theorem, one computes

$$\begin{aligned}2kV_w(B_s\cap\Sigma) &= \int_{B_s\cap\Sigma} w\Delta_\Sigma\|x\|^2 \\ &= \int_{B_s\cap\Sigma}\|x\|^2\Delta_\Sigma w + \int_{B_s\cap\Sigma}\operatorname{div}_\Sigma\big(w\nabla_\Sigma\|x\|^2 - \|x\|^2\nabla_\Sigma w\big) \\ &= \int_{B_s\cap\Sigma}\|x\|^2\Delta_\Sigma w + \int_{\partial B_s\cap\Sigma}\big\langle w\nabla_\Sigma\|x\|^2, N\big\rangle - \int_{\partial B_s\cap\Sigma}\big\langle \|x\|^2\nabla_\Sigma w, N\big\rangle \\ &= \int_{B_s\cap\Sigma}\|x\|^2\Delta_\Sigma w + \int_{\partial B_s\cap\Sigma} 2w\|x^\top\| - s^2\int_{\partial B_s\cap\Sigma}\langle\nabla_\Sigma w, N\rangle \\ &= \int_{B_s\cap\Sigma}\|x\|^2\Delta_\Sigma w + 2\int_{\partial B_s\cap\Sigma} w\|x^\top\| - s^2\int_{B_s\cap\Sigma}\Delta_\Sigma w.\end{aligned}$$

Now all that's left is to follow the same steps as in the proof for the monotonicity formula.

□

References

[1] T. H. Colding and W. P. Minicozzi II, *A course in minimal surfaces*, vol. 121, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2011, DOI: 10.1090/gsm/121.

[2] R. Osserman, *The convex hull property of immersed manifolds*, J. Differential Geometry 6 (1971/72), pp. 267–270, ISSN: 0022-040X.