# Notes on Minimal Surfaces

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## 1 (5.26.23) Bernstein's Theorem

Follows 1.4 - 1.5 of [CM11].

**Theorem 1.0.1** (Bernstein)**.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation, then its graph must be an affine plane.*

*Remark* 1.0.2. It's interesting to compare this statement to *Liouville theorem*, namely bounded (or sublinear) entire harmonic functions are constant.

### 1.1 Preliminaries

For $\Sigma^2 \subset \mathbb{R}^3$ be orientable and $\nu$ be a choice of unit normal on $\Sigma$. We will define two maps, which will be identified with each other [Figure 1]. The first

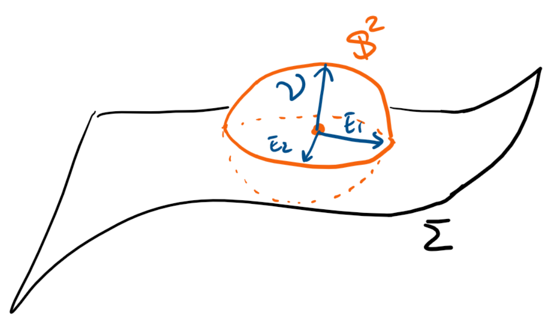


Figure 1: Weingarten map is the differential of Gauss map

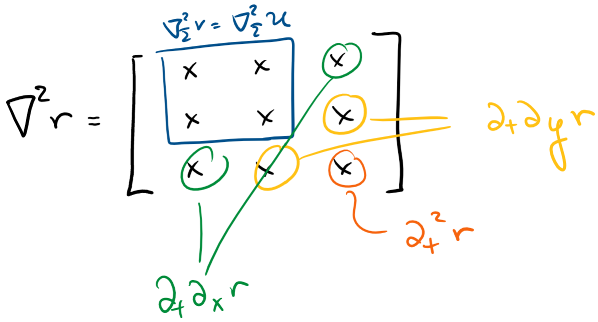


Figure 2: The full Hessian of $r$

map is the *Weingarten map*,

$$\begin{aligned} T\Sigma &\to T\Sigma \\ X &\mapsto \nabla_X \nu. \end{aligned}$$

Note that changing this to the $(0,2)$ version gives the second fundamental form

$$\mathrm{II}(X,Y) := \langle \nabla_X \nu, Y\rangle.$$

In particular, the Weingarten map is symmetric, real-valued so it diagonalizes with eigenvalues $\kappa_1, \kappa_2$. The second map is the differential of the *Gauss map*,

$$\begin{aligned} d\nu : T\Sigma &\to T\mathbb{S}^2 \\ X &\mapsto d\nu(X). \end{aligned}$$

The two maps can be identified since any orthonormal frame $E_1, E_2$ on $\Sigma$ can be carried to one on $\mathbb{S}^2$, so there is no point in distinguishing the codomain between $T\Sigma$ or $T\mathbb{S}^2$. Furthermore assuming $\Sigma$ is minimal forces $\kappa_2 = -\kappa_1$ and gives anti-conformality of the Gauss map,

$$|d\nu|^2 = |\mathrm{II}|^2 = \kappa_1^2 + \kappa_2^2 = -2\kappa_1\kappa_2 = -2\det(d\nu). \tag{1}$$

We briefly remark that $\det(d\nu)$ is a common definition of *Gauss curvature*.

We observe if $\Sigma$ is given as the regular level set of a function $r : \mathbb{R}^3 \to \mathbb{R}$, then its second fundamental form is proportional the surface Hessian. Recall for $X, Y \in \mathfrak{X}(\Sigma)$,

$$\nabla^2_\Sigma r(X,Y) := \langle \nabla_X \nabla r, Y\rangle.$$

Since the gradient is perpendicular to level sets, the claim follows

$$\mathrm{II}(X,Y) = \langle \nabla_X \frac{\nabla u}{|\nabla u|}, Y\rangle = \frac{\nabla^2_\Sigma u(X,Y)}{|\nabla u|} + \underbrace{X|\nabla u|\langle \nabla u, Y\rangle}_{=0}.$$

We will apply this observation to Bernstein's theorem via the following procedure which turns any graph into a level set. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a solution to the minimal surface equation, and consider $\Sigma = \text{graph}\, u \subset \mathbb{R}^3$ with the induced metric. Consider the signed distance function $r$ in a neighborhood of $\Sigma$,

$$\begin{aligned} r : \Sigma \times (-\epsilon, \epsilon) &\to \mathbb{R} \\ (x,t) &\mapsto u(x) - t. \end{aligned}$$

We compute the norm of the gradient of $r$ and the Hessian of $r$ as

$$|\nabla r| = \sqrt{(\partial_t r)^2 + |\nabla_\Sigma r|^2} = \sqrt{1 + |\nabla u|^2},$$
$$\nabla^2_\Sigma r = \nabla^2_\Sigma u.$$

The Hessian identity follows since all derivatives in the direction of $\Sigma$ fall onto $u$ [Figure 2]. Therefore, the second fundamental form can be expressed purely in terms of $u$,

$$\Pi = \frac{\nabla^2_\Sigma r}{|\nabla r|} = \frac{\nabla^2_\Sigma u}{\sqrt{1+|\nabla u|^2}}.$$

In particular, if the second fundamental form vanishes, then so must the Hessian, so $u$ will graph an affine plane.

## 1.2 Logarithm Cutoff

We begin our quest of showing $\mathrm{II} \equiv 0$ on $\Sigma := \text{graph}\, u$ by showing that the total curvature is bounded by the energy of any cutoff function.

**Lemma 1.2.1.** *Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a solution to the minimal surface equation. For any non-negative, Lipschitz*[1] *function $\eta$ with support contained in $\Omega \times \mathbb{R}$,*

$$\int_\Sigma \eta^2 |II|^2 \le C \int_\Sigma |\nabla_\Sigma \eta|^2.$$

[1] It is helpful to recall Rademacher's theorem, which tells us that Lipschitz functions are differentiable almost everywhere.

*Proof.* Let $\omega$ be the area form on $\mathbb{S}^2$ and consider the upper hemisphere. Consider the 1-form $\alpha$ such that $d\alpha = \omega$ on the upper hemisphere. Equation 1 implies

$$|\mathrm{II}|^2 d\Sigma = -2\det(d\nu)d\Sigma = 2\nu^*\omega = 2d\nu^*\alpha.$$

Furthermore, in local coordinates we have $(\nu^*\alpha)_i = (d\nu)_i^j \alpha_j$, so Cauchy-Schwarz implies

$$|\nu^*\alpha| \leq C|\mathrm{II}|,$$

with $C = C(\alpha)$. In total,

$$\int_\Sigma \eta^2 |\mathrm{II}|^2 d\Sigma = 2\int_\Sigma \eta^2 d\nu^*\alpha = \underbrace{-4\int_\Sigma \eta d\eta \wedge \nu^*\alpha}_{\text{Stokes \& } \eta^2 \text{ vanishes on } \partial\Sigma}$$

$$\leq 4C\int_\Sigma \eta|\nabla_\Sigma \eta||\mathrm{II}|d\Sigma \leq 4C\left(\int_\Sigma \eta^2|\mathrm{II}|^2 d\Sigma\right)^{\frac{1}{2}}\left(\int_\Sigma |\nabla_\Sigma \eta|^2 d\Sigma\right)^{\frac{1}{2}},$$

and so we reincorporate to get

$$\int_\Sigma \eta^2|\mathrm{II}|^2 d\Sigma \leq 16C^2\int_\Sigma |\nabla_\Sigma \eta|^2 d\Sigma.$$

■

*Remark* 1.2.2. I'm not convinced that non-negativity of $\eta$ is used in any meaningful way in the above proof. This assumption can probably be dropped.

In light of Lemma 1.2.1, game now is to find a sequence of non-negative Lipschitz cutoff functions $\eta_N$ tending to 1, with energy tending to 0. Let us first work heuristically. Define radial cutoff functions [Figure 3] for $r = |x|$,

$$\eta_N(r) := \begin{cases} 1 & r \leq e^N \\ 2 - \frac{\log(r)}{N} & e^N < r \leq e^{2N} \\ 0 & e^{2N} < r. \end{cases}$$

Observe $\eta_N \to 1$ as $N \to \infty$. We compute $|\nabla \eta_N| = \frac{1}{Nr}$,[2] and by co-area formula we compute

$$\int_{\mathbb{R}^2} |\nabla \eta_N|^2 = \int_0^\infty \int_{\partial B_r} |\nabla \eta_N|^2 d\sigma dr = \int_{e^N}^{e^{2N}} \frac{2\pi r}{(Nr)^2}dr = \frac{2\pi}{N}.$$

[2] In fact, you probably start at this and define $\eta$ from here.

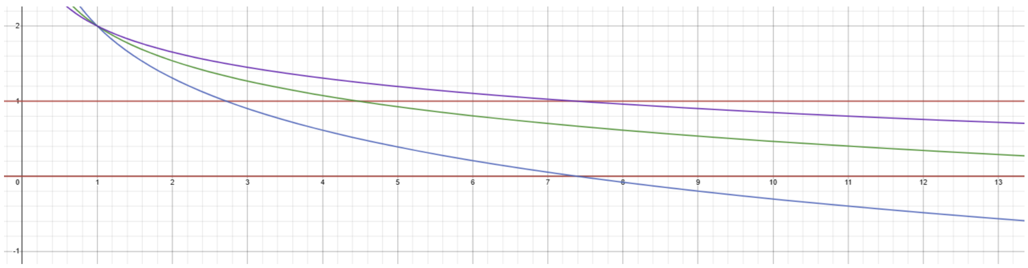

Figure 3: $\eta_N$ for $N = 1, 1.5, 2$

In particular, the energy of $\eta_N$ vanishes as $N \to \infty$. The same computation holds with *linear perimeter growth*

$$\text{Length}(\partial B_r) \leq Cr,$$

as we may basically repeat the above argument

$$\int_\Sigma |\nabla \eta_N|^2 = \int_0^\infty \int_{\partial B_r} |\nabla \eta_N|^2 d\sigma dr = \int_{e^N}^{e^{2N}} \frac{\text{Length}(\partial B_r)}{(Nr)^2} dr \leq \frac{C}{N}.$$

However, what's important is that the same conclusion holds under the assumption of *quadratic area growth*

$$\text{Area}(B_r) \leq Cr^2.$$

This is important since by a calibration argument (Corollary 1.2 of [CM11]), a minimal surface $\Sigma^2 \subset \mathbb{R}^3$ will always obey a quadratic area growth (with $C = 2\pi$). To prove the energy bound under this assumption, first observe $|\nabla \eta_N|$ is monotonically decreasing, so

$$\sup_{B_{e^k} \setminus B_{e^{k-1}}} |\nabla_N \eta|^2 = |\nabla \eta_N|^2 \Big|_{\partial B_{e^{k-1}}} = N^{-2} e^{2-2k}.$$

We break up the "middle section" into concentric annuli and compute

$$\begin{aligned} \int_\Sigma |\nabla \eta_N|^2 &\leq \sum_{k=N+1}^{2N} \int_{B_{e^k} \setminus B_{e^{k-1}}} N^{-2} e^{2-2k} d\sigma \\ &\leq \sum_{k=N+1}^{2N} N^{-2} e^{2-2k} \text{Area}(B_{e^k} \setminus B_{e^{k-1}}) \leq \sum_{k=N+1}^{2N} C N^{-2} e^2 = \frac{Ce^2}{N}. \end{aligned}$$

*Remark* 1.2.3. Let's take a second to summarize what happened. The energy integrand decays like $r^{-2}$, while the domain grows like $r^2$. To get the desired decay rate, you associate a constant $N$ with the energy integrand, which pops out as $N^{-2}$. This combats the linear $N$ that pops out of the sum over the annuli, leaving a final rate of $N^{-1}$.

*Remark* 1.2.4. This whole heuristic can obviously be sharpened. Two immediate directions are replacing Cauchy-Schwarz with Hölder in Lemma 1.2.1, and requiring a decay on the energy of $\eta_N$ of $N^{-\alpha}$ for any $\alpha > 0$. Generalizing in these directions is the content of Chapter 2.

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Inspired by these heuristic computations, we perform a similar logarithmic cutoff trick to conclude Bernstein's theorem.

**Corollary 1.2.5.** *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$ and $\Omega$ contains a ball of radius $\kappa R$ centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}R} \cap \Sigma} |II|^2 \leq \frac{C}{\log \kappa}.$$

*Remark* 1.2.6. Note that the parameter $R$ is needed as we require $\kappa > 1$ for taking log. But for Bernstein purposes, we can think of fixing $R = 1$ and $\kappa \to \infty$.

*Proof.* Define $\eta : \mathbb{R}^3 \to \mathbb{R}$ with support contained in $B_{\kappa R}$. Again for $r = |x|$, define

$$\eta(r) := \begin{cases} 1 & r \leq \sqrt{\kappa}R \\ 2 - \frac{2\log(\frac{r}{R})}{\log \kappa} & \sqrt{\kappa}R < r \leq \kappa R \\ 0 & \kappa R < r. \end{cases}$$

We compute $|\nabla_\Sigma \eta| \leq \frac{2}{r \log \kappa}$, and assuming for simplicity $\log \sqrt{\kappa} = \log \kappa / 2$ is an integer,

$$\int_{B_{\sqrt{\kappa}R}\cap\Sigma} |\mathrm{II}|^2 \leq \int_\Sigma \eta^2|\mathrm{II}|^2 \leq C\int_\Sigma |\nabla_\Sigma \eta|^2 \leq \frac{4C}{(\log\kappa)^2} \underbrace{\int_{B_{\kappa R}\cap\Sigma} r^{-2}dr}_{\text{quadratic decay}}$$

$$\leq \frac{4C}{(\log\kappa)^2}\sum_{k=\log\sqrt{\kappa}}^{\log\kappa}\int_{B_{e^kR}\setminus B_{e^{k-1}R}\cap\Sigma} r^{-2}dr$$

$$\leq \frac{4C}{(\log\kappa)^2}\sum_{k=\log\sqrt{\kappa}}^{\log\kappa}(e^{k-1}R)^{-2}\cdot \underbrace{2\pi(e^kR)^2}_{\text{quadratic area growth}}$$

$$= \frac{4C}{(\log\kappa)^2}\sum_{k=\log\sqrt{\kappa}}^{\log\kappa} 2e^2\pi = \frac{4\pi e^2 C}{\log\kappa}.$$

■

# 2 (6.16.23) PDE Aspects of Stability

Follows 1.8.3 of [CM11]. It is interesting to compare the contents of this section with [Cha84, Ch 1.5].

## 2.1 Principal Eigenvalue of Stability

In this section, we mimic some classical computations with the Laplacian with the stability operator. Recall the (linear, elliptic) *stability operator*

$$L\eta := \Delta_\Sigma\eta + |A|^2\eta + \mathrm{Ric}(\nu,\nu)\eta,$$

for $\Sigma^{n-1} \subset M^n$ a stable 2-sided minimal hypersurface with unit normal $\nu$ and $X = \eta\nu$. The convention on the Laplacian is that that its eigenvalues are *negative*.

Recall that *stability* of a minimal hypersuface $\Sigma$ is the requirement that the operator $L$ is negative semi-definite (or equivalently, $-L$ is positive semi-definite) over all subdomains $\Omega \subset \Sigma$. Equivalently, for fixed $\Omega$, we phrase stability as non-negativity of the principal (Dirichlet) eigenvalue $\lambda_1$, defined as Rayleigh quotient

$$\lambda_1 := \inf\left\{\frac{-\int \eta L\eta}{\int \eta^2} : \eta \in C_0^\infty(\Omega)\right\}$$
$$= \inf\{-\int \eta L\eta : \eta \in C_0^\infty(\Omega), \int \eta^2 = 1\}. \tag{2}$$

Here and throughout, all integrals will be taken over a bounded domain $\Omega \subset \Sigma$. Thus, our goal is to understand when $\lambda_1 \geq 0$.

We first show (2) admits a weak formulation, and run the standard elliptic machinery. By integration by parts, for every $\eta \in C_0^\infty(\Omega)$

$$\int \Delta_\Sigma \eta = -\int |\nabla_\Sigma \eta|^2, \tag{3}$$

thereby weakly turning the $2^{\text{nd}}$ order term of $L$ into a $1^{\text{st}}$ order one. Therefore, consider the problem of minimizing the Rayleigh quotient

$$I := \inf\left\{\frac{\int |\nabla_\Sigma \eta|^2 - |A|^2\eta^2 - \text{Ric}(N,N)\eta^2}{\int \eta^2} : \eta \in W_0^{1,2}(\Omega)\right\}$$

over the larger function space $W_0^{1,2}(\Omega)$.

**Lemma 2.1.1.** *In the notation above, $\lambda_1 = I$. Furthermore, if a weak solution $u \in W^{1,2}(\Omega)$ achieves equality*

$$\lambda_1 = \frac{\int |\nabla_\Sigma u|^2 - |A|^2u^2 - Ric(\nu,\nu)u^2}{\int u^2},$$

*then automatically $u \in C_0^\infty(\Omega)$ and $-Lu = \lambda_1 u$.*

*Proof.* It is clear that $\lambda_1 \geq I$ by (3) and $C_0^\infty(\Omega) \subset W_0^{1,2}(\Omega)$ (since taking infimum over a larger space could only possibly give a lower value). For simplicity, denote the $0^{\text{th}}$ order term of $L$ by

$$V(x) := |A|^2(x) + \text{Ric}(\nu,\nu)(x).$$

For the other direction, consider $\{\eta_j\} \subset W_0^{1,2}(\Omega)$ a minimizing sequence to $I$ so that

$$I + \frac{1}{j} \geq \frac{\int |\nabla_\Sigma \eta_j|^2 - V\eta_j^2}{\int \eta_j^2}. \tag{4}$$

Observe the left hand side of (4) is preserved under scaling ($\eta_j \mapsto c_j\eta_j$), so assume $\int \eta_j^2 = 1$. Since the sequence $\{\eta_j\}$ is minimizing, it is bounded; sequential Banach-Alaoglu implies that it weakly converges to a function $\eta \in W_0^{1,2}$. Recall that inclusion $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ is compact by Rellich for $p \leq n$[3] and Morrey for $p > n$ as in [Eva10, Ch 5.7]. Therefore, the convergence to $\eta$ is strong in $L^2$; in particular

$$\liminf \int \eta_j^2 = \int \eta^2 = 1.$$

It follows from taking liminf on both sides of (4) that

$$\begin{aligned} I &\geq \liminf \int |\nabla_\Sigma \eta_j|^2 - \liminf \int V\eta_j^2 \\ &\geq \int |\nabla_\Sigma \eta|^2 - \int V\eta^2. \end{aligned}$$

In the second inequality, the first term follows from weak lower semi-continuity of energy. For the second terms first note that $V \in L^\infty(\Omega)$ since it is continuous up to the boundary, and by Hölder $V\eta_j^2$ and $V\eta^2$ are integrable. The second term thus follows since $\eta_j \xrightarrow{L^2} \eta$, then $\eta_j^2 \xrightarrow{L^1} \eta^2$. The continuous evaluation pairing $L^1(\Omega)^* \times L^1(\Omega) \to \mathbb{R}$ can be identified by Riesz representation theorem as

$$(V, \eta) \mapsto \int V\eta$$

for $V \in L^\infty(\Omega)$ and $\eta \in L^1(\Omega)$. Equality follows by weak convergence of $\eta_j^2 \to \eta^2$. In particular, definition of $I$ forces $\eta \in W_0^{1,2}(\Omega)$ to be a minimizer, i.e.

$$I = \int |\nabla_\Sigma \eta|^2 - V\eta^2.$$

We now apply a Dirichlet principle to show such a minimizer satisfies $L\eta = I\eta$ weakly; smoothness of $\eta$ is immediate from elliptic regularity. Consider a perturbation of (the weak form of) $L$ at the minimizer $\eta$ by $\psi \in C_0^\infty(\Omega)$;

[3] For the $p = n$ case, we need to argue with the contravariant $L^p$ inclusion on bounded domains. We know that $W^{1,n} \subset W^{1,q}$ for each $q \in [1, n)$ by applying the above to a function and its weak first derivative. We then use Rellich to get a compact embedding of $W^{1,q} \subset L^n$ for $q$ close enough to $n$. Close enough means choosing $q < n$ to solve the inequality $n^2 < 2nq$, which comes from looking at $n < q^* = \frac{nq}{n-q}$.

by density, the same analysis will hold for $\psi \in W_0^{1,2}(\Omega)$. As minimizers are critical points,

$$0 = \frac{d}{dt}\bigg|_{t=0} \frac{1}{2}\int |\nabla_\Sigma(\eta + t\psi)|^2 - V(\eta + t\psi)^2 = \int \langle \nabla_\Sigma \eta, \nabla_\Sigma \psi \rangle - V\eta\psi. \quad (5)$$

The RHS of (5) is the statement that $\eta$ weakly solves $L$. Now, restrict to variations $\psi \in W_0^{1,2}(\Omega)$ such that

$$\int \eta\psi = 0. \quad (6)$$

Given $\phi \in W_0^{1,2}(\Omega)$, set

$$\psi := \phi - \eta \underbrace{\left(\int \eta\phi\right)}_{\text{a constant!}}.$$

$\psi$ satisfies condition (6) as

$$\int \eta\psi = \int \eta\phi - \underbrace{\left(\int \eta^2\right)}_{=1}\left(\int \eta\phi\right) = 0.$$

Plugging in this choice of $\psi$ into (5),

$$\begin{aligned} 0 &= \int \langle \nabla_\Sigma \eta, \nabla_\Sigma \psi \rangle - V\eta\psi \\ &= \int \langle \nabla_\Sigma \eta, \nabla_\Sigma (\phi - \eta\left(\int \eta\phi\right)) \rangle - V\eta(\phi - \eta\left(\int \eta\phi\right)) \\ &= \int \langle \nabla_\Sigma \eta, \nabla_\Sigma \phi \rangle - \left(\int \eta\phi\right)|\nabla_\Sigma \eta|^2 - V\eta\phi + V\eta^2\left(\int \eta\phi\right). \end{aligned}$$

In particular, we conclude the proof since

$$\int \langle \nabla_\Sigma \eta, \nabla_\Sigma \phi \rangle - V\eta\phi = \left(\int \eta\phi\right)\int |\nabla_\Sigma \eta|^2 - V\eta^2 = \left(\int \eta\phi\right) I.$$

■

Combining the previous result with Harnack's inequality, we get an analog of *Courant's nodal domain theorem.* In particular, the first eigenfunction of $L$ has multiplicity one.

**Lemma 2.1.2.** *If $u$ is a smooth function on $\Omega$ that vanishes on $\partial\Omega$ and $Lu = -\lambda_1 u$, then $u$ cannot change sign.*

*Proof.* Assume $u \not\equiv 0$. It is easy to show $|u| \geq 0$ is also a $W_0^{1,2}$ solution, and Harnack implies that $|u| > 0$. ∎

## 2.2 Minimal Graphs are Stable

The following is due to Barta.

**Lemma 2.2.1.** *Let $\Sigma$ be a 2-sided minimal hypersurface, and $\Omega \subset \Sigma$. If there exists a positive solution $u > 0$ on $\Omega$ to $Lu = 0$, then $\Omega$ is stable.*

*Proof.* Positivity of $u$ allows us to consider $\log u$, and we make a computation $\Delta_\Sigma \log u$. Let $i$ run through geodesic normal coordinates at a point on $\Sigma$,

$$\begin{aligned}\Delta_\Sigma \log u &= \sum_i \partial_i \partial_i \log u = \sum_i \partial_i \left(\frac{u_i}{u}\right) = \sum_i -\left(\frac{u_i}{u}\right)^2 + \frac{u_{ii}}{u} \\ &= \sum_i -(\log u)_i^2 + \frac{\Delta_\Sigma u}{u} = -|\nabla_\Sigma \log u|^2 - V.\end{aligned}$$

The last equality uses $Lu = 0$ on the second term; by positivity of $u$, all denominators are valid. For $f \in C_0^\infty(\Omega)$, we see

$$\begin{aligned}\int f^2 V + \int f^2 |\nabla_\Sigma \log u|^2 &= -\int f^2 \Delta_\Sigma \log u \\ &= \int \nabla_\Sigma f^2 \cdot \nabla_\Sigma \log u \\ &= 2\int f \nabla_\Sigma f \cdot \nabla_\Sigma \log u \\ &\leq 2\int |f||\nabla_\Sigma f||\nabla_\Sigma \log u| \\ &\leq \int |\nabla_\Sigma f|^2 + \int f^2 |\nabla_\Sigma \log u|^2\end{aligned}$$

with the last inequality using $|a \cdot b| \leq \frac{|a|^2}{2} + \frac{|b|^2}{2}$. From integration by parts follows stability. ■

As a corollary, we show minimal graphs are automatically stable, following [Sun16, Cor 6.1].

**Corollary 2.2.2.** *Minimal graphs are stable.*

*Proof.* For $\Sigma = \text{graph } u$ with unit normal $\nu$, we claim

$$\langle \nu, \partial_z \rangle = \langle \frac{(-u_x, u_y, 1)}{\sqrt{1 + |\nabla u|^2}}, \partial_z \rangle = \frac{1}{\sqrt{1 + |\nabla u|^2}}$$

is a (positive) Jacobi field. Since the ambient manifold is $\mathbb{R}^3$, the Ricci term in the stability operator vanishes, and it suffices to show

$$\Delta_\Sigma \langle \nu, \partial_z \rangle = -|A|^2 \langle \nu, \partial_z \rangle.$$

Let $i, j \in \{1, 2\}$ run through geodesic normal coordinates at a point on $\Sigma$. We pause to collect some basic observations used in the subsequent computation.

1. $\partial_z$ is a parallel vector field (i.e. $\nabla \partial_z \equiv 0$).
2. Recall the Bianchi identity for $A_{ij} := \langle \nabla_i \nu, \partial_j \rangle$,

$$\begin{aligned} A_{ij,i} &= \partial_i \langle \nabla_i \nu, \partial_j \rangle \\ &= \langle \nabla_i \nabla_i \nu, \partial_j \rangle + \langle \nabla_i \nu, \nabla_i \partial_j \rangle \\ &= \langle \nabla_j \nabla_i \nu, \partial_i \rangle + \langle \nabla_i \nu, \nabla_j \partial_i \rangle \\ &= \partial_j \langle \nabla_i \nu, \partial_i \rangle = A_{ii,j}. \end{aligned}$$

   In the third equality, the first term follows from symmetry of the Hessian, as we may locally solve $\nabla_i \nu = \nabla u$ for a function $u$. The second follows from the torsion-free property of the connection.

3. Christoffel symbols vanish *along the surface* $\Sigma$, so

$$\nabla_i \partial_j = \langle \nabla_i \partial_j, \nu \rangle \nu = -A_{ij} \nu.$$

We use the above facts to compute

$$
\begin{aligned}
\Delta_\Sigma \langle \nu, \partial_z \rangle &= \sum_i \partial_i \partial_i \langle \nu, \partial_z \rangle = \sum_i \partial_i \langle \nabla_i \nu, \partial_z \rangle \\
&= \sum_i \partial_i \langle \sum_j A_{ij} \partial_j, \partial_z \rangle \\
&= \sum_{i,j} \langle A_{ij,i} \partial_j, \partial_z \rangle + \sum_{i,j} \langle A_{ij} \nabla_i \partial_j, \partial_z \rangle \\
&= \underbrace{\sum_{i,j} \langle A_{ii,j} \partial_j, \partial_z \rangle}_{=0} - |A|^2 \langle \nu, \partial_z \rangle \\
&= -|A|^2 \langle \nu, \partial_z \rangle.
\end{aligned}
$$

The last equality follows from minimality of $\Sigma$,

$$
\sum_{i,j} A_{ii,j} = \sum_j \partial_j \left( \sum_i \langle \nabla_i \nu, \partial_i \rangle \right) = \sum_j \partial_j H \equiv 0.
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■

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