Moving Frames

Dedicated to *Moving Frames* of summer '23.

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1 Conventions

We always work on Riemannian manifolds with the Levi-Civita connection.

- No index balancing - we'll use all lower indices and explicit summations.
- E_i's denote (orthonormal) frames, Θ_i's denote (their dual) coframes, ω_{ij}'s denote the connection 1-forms, and Ω_{ij}'s denote the curvature 2-forms.
- Tildes will denote the "new" (the unknown), while no tildes denote the "old" (the known).
- For non-products, Latin will be used as a general index.

  For products, we use both Latin and Greek alphabets to separate out individual factors. In such cases, we will use p, q to denote indices which run over both Latin and Greek; in particular, do **not** think of p, q as Latin indices.[1]
- We define the connection 1-forms as

  $$\nabla_X E_i = \sum_j \omega_{ij}(X) E_j,$$

  and curvature 2-forms as

  $$R(X, Y)E_i = \sum_j \Omega_{ij}(X, Y) E_j,$$

  for R the $(1, 3)$ curvature tensor. Thus, the structure equations [Propositions 3.2 and 3.3] are

  - **(C1)** $d\Theta_{ij} = \sum_j \omega_{ij} \wedge \Theta_j$.
  - **(C2)** $\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}$.

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[1]Sorry, but there's no canonical 3$^{\text{rd}}$ alphabet.

Both ω_{ij} and Ω_{ij} are anti-symmetric in the indices with respect to an orthonormal frame [Proposition 3.4].

- Under this convention, the negative of the scalar curvature is the trace of both indices. Namely, if R is the scalar curvature, then

$$-R = \sum_{i,j} \Omega_{ij}(E_i, E_j).$$

2 Computations

We open with an easy example to illustrate the basic framework.

Proposition 2.1. *Show that the curvature tensors are additive under a product metric.*

Proof. Let $\{\Theta_i\}$ (resp. $\{\Theta_\alpha\}$) denote a coframe on M (resp. N). Then, the induced coframe on $(M \times N, g_M + g_N)$ is given by $\{\tilde{\Theta}_i, \tilde{\Theta}_\alpha\}$, where $\tilde{\Theta}_i = \Theta_i$ and $\tilde{\Theta}_\alpha = \Theta_\alpha$.

We use first structure equation to derive the new connection 1-forms. We compute from the M factor

$$\sum_j \tilde{\omega}_{ij} \wedge \tilde{\Theta}_j + \sum_\alpha \tilde{\omega}_{i\alpha} \wedge \tilde{\Theta}_\alpha = d\tilde{\Theta}_i = d\Theta_i = \underbrace{\sum_j \omega_{ij} \wedge \Theta_j}_{\text{(C1) on } M} = \sum_j \omega_{ij} \wedge \tilde{\Theta}_j.$$

A similar computation on the N factor gives

$$\sum_i \tilde{\omega}_{\alpha i} \wedge \tilde{\Theta}_i + \sum_\beta \tilde{\omega}_{\alpha\beta} \wedge \tilde{\Theta}_\beta = d\tilde{\Theta}_\alpha = \sum_\beta \omega_{\alpha\beta} \wedge \tilde{\Theta}_\beta.$$

From these two equations, we guess the new connection 1-forms as

$$\tilde{\omega}_{pq} = \begin{cases} & (p,q) \\ \hline \omega^M & \text{Latin} \\ \omega^N & \text{Greek} \\ 0 & \text{Bilingual} \end{cases} \iff \tilde{\omega} = \left(\begin{array}{c|c} \omega^M & 0 \\ \hline 0 & \omega^N \end{array} \right).$$

By uniqueness of the connection 1-forms, this guess is correct. We move on to compute the new connection 2-form from the second structure equation. A moment's glance at (C2) reveals that there are no mixed terms $\tilde{\Omega}_{i\alpha}$, as $\tilde{\omega}$'s have no mixed terms. Another moment's glance reveals

$$\tilde{\Omega} = \left(\begin{array}{c|c} \Omega^M & 0 \\ \hline 0 & \Omega^N \end{array} \right),$$

and so the assertion follows. ■

Let's move on to a harder example that reveals some general difficulties.

Proposition 2.2. *Show that the scalar curvature $\tilde{R}$ of a warped product metric $g_B + f^2 g_F$ on $B \times_f F$ for $f : B \to \mathbb{R}^{>0}$ is*

$$\tilde{R} = R^B + \frac{R^F}{f^2} - n(n-1)\left(\frac{|\nabla f|}{f}\right)^2 - \frac{2n\Delta f}{f},$$

for $n = \dim F$, and R^B (resp. R^F) the scalar curvatures on B (resp. F).

Proof. Let $\{E_i\}$ (resp. $\{E_\alpha\}$) denote a frame on B (resp. F). Let $\{\Theta_i\}$ (resp. $\{\Theta_\alpha\}$) denote the dual coframe on B (resp. F). Then, the induced coframe on $B \times_f F$ is given by $\{\tilde{\Theta}_i, \tilde{\Theta}_\alpha\}$, where $\tilde{\Theta}_i = \Theta_i$ and $\tilde{\Theta}_\alpha = f\Theta_\alpha$.

We compute (C1) from the B factor

$$\sum_i \tilde{\omega}_{ij} \wedge \tilde{\Theta}_j + \sum_\alpha \tilde{\omega}_{i\alpha} \wedge \tilde{\Theta}_\alpha = d\tilde{\Theta}_i = d\Theta_i = \underbrace{\sum_j \omega_{ij}\Theta_j}_{\text{(C1) on } B} = \sum_j \omega_{ij}\tilde{\Theta}_j. \tag{1}$$

From this equation alone, we may be tempted to guess $\tilde{\omega}_{ij} = \omega_{ij}$ and $\tilde{\omega}_{i\alpha} = 0$. The second guess is wrong - we get mixed terms in the new connection 1-forms. This is not surprising, as we would otherwise run the previous argument to get that the curvature tensors of a warped product are additive (and independent of f). If this were true, we would never care about the geometry of warped products. The newly revealed phenomenon is the following fact,

$$\ker(- \wedge \tilde{\Theta}_\alpha) = \text{ multiples of } \tilde{\Theta}_\alpha.$$

In other words, we only pin down $\tilde{\omega}_\alpha$ up to a multiple of $\tilde{\Theta}_\alpha$, i.e. for some function ϕ_i,

$$\tilde{\omega}_{i\alpha} = \phi_i \tilde{\Theta}_\alpha.$$

The exact function ϕ_i will be pinned down with (C1) from the F factor. We compute

$$\begin{aligned}
\sum_i \tilde{\omega}_{\alpha i} \wedge \tilde{\Theta}_i + \sum_\beta \tilde{\omega}_{\alpha\beta} \wedge \tilde{\Theta}_\beta = d\tilde{\Theta}_\alpha &= d(f\Theta_\alpha) = df \wedge \Theta_\alpha + f d\Theta_\alpha \\
&= \sum_i (E_i f)\Theta_i \wedge \Theta_\alpha + f \underbrace{\sum_\beta \omega_{\alpha\beta} \wedge \Theta_\beta}_{\text{(C1) on } F} \\
&= \sum_i -(E_i f)\Theta_\alpha \wedge \tilde{\Theta}_i + \sum_\beta \omega_{\alpha\beta} \wedge \tilde{\Theta}_\beta
\end{aligned} \tag{2}$$

Therefore, $\tilde{\omega}_{\alpha i} = -(E_i f)\Theta_\alpha$ and so $\phi_i = \frac{E_i f}{f}$. Equations (1) and (2) together give the new connection 1-forms as

$$\tilde{\omega}_{pq} = \begin{cases} & (p,q) \\ \hline \omega_{ij} & (i,j) \\ \omega_{\alpha\beta} & (\alpha,\beta) \\ (E_i f)\Theta_\alpha & (i,\alpha) \end{cases} \iff \tilde{\omega} = \left(\begin{array}{c|c} \omega^B & \ldots \tilde{\omega}_{i\alpha} = (E_i f)\Theta_\alpha \ldots \\ \hline \vdots & \\ \tilde{\omega}_{\alpha i} = -(E_i f)\Theta_\alpha & \omega^F \\ \vdots & \end{array}\right).$$

We move on the compute (C2) with Latin indices

$$\begin{aligned}
\tilde{\Omega}_{ij} &= d\tilde{\omega}_{ij} - \sum_p \tilde{\omega}_{ip} \wedge \tilde{\omega}_{pj} \\
&= \underbrace{d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}}_{\text{(C2) on } B} - \sum_\alpha \tilde{\omega}_{i\alpha} \wedge \tilde{\omega}_{\alpha j} \\
&= \Omega_{ij} + \underbrace{\sum_p (E_i f)\Theta_\alpha \wedge (E_j f)\Theta_\alpha}_{=0} \\
&= \Omega_{ij},
\end{aligned}$$

with Greek indices

$$\begin{aligned}
\tilde{\Omega}_{\alpha\beta} &= d\tilde{\omega}_{\alpha\beta} - \sum_p \tilde{\omega}_{\alpha p} \wedge \tilde{\omega}_{p\beta} \\
&= \underbrace{d\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}}_{\text{(C2) on } F} - \sum_i \tilde{\omega}_{\alpha i} \wedge \tilde{\omega}_{i\beta} \\
&= \Omega_{\alpha\beta} + \sum_i (E_i f)^2 \Theta_\alpha \wedge \Theta_\beta \\
&= \Omega_{\alpha\beta} + |\nabla f|^2 \Theta_\alpha \wedge \Theta_\beta,
\end{aligned}$$

and with bilingual indices

$$\begin{aligned}
\tilde{\Omega}_{i\alpha} &= d\omega_{i\alpha} - \sum_p \tilde{\omega}_{ip} \wedge \tilde{\omega}_{p\alpha} \\
&= d((E_i f)\Theta_\alpha) - \sum_j \tilde{\omega}_{ij} \wedge \tilde{\omega}_{j\alpha} - \sum_\beta \tilde{\omega}_{i\alpha} \wedge \tilde{\omega}_{\beta\alpha} \\
&= (E_j E_i f)\Theta_j \wedge \Theta_\alpha + \underbrace{(E_i f) d\Theta_\alpha - \sum_\beta (E_i f)\Theta_\beta \wedge \omega_{\beta\alpha}}_{=0} - \sum_j \omega_{ij} \wedge (E_j f)\Theta_\alpha \\
&= \underbrace{(\sum_j (E_j E_i)\Theta_j - (E_j f)\omega_{ij})}_{\text{Proposition 3.1}} \wedge \Theta_\alpha \\
&= \nabla^2 f(-, E_i) \wedge \Theta_\alpha.
\end{aligned}$$

Altogether, we have

$$\tilde{\Omega}_{pq} = \begin{cases} & (p,q) \\ \hline \Omega_{ij} & (i,j) \\ \Omega_{\alpha\beta} + |\nabla f|^2 \Theta_\alpha \wedge \Theta_\beta & (\alpha,\beta) \\ \nabla^2 f(-, E_i) \wedge \Theta_\alpha & (i,\alpha) \end{cases} \iff$$

$$\tilde{\Omega}_{pq} = \left(\begin{array}{c|c} \Omega^B_{ij} & \ldots \tilde{\Omega}_{i\alpha} = \nabla^2 f(-, E_i) \wedge \Theta_\alpha \ldots \\ \hline \vdots & \\ \tilde{\Omega}_{\alpha i} = -\nabla^2 f(-, E_i) \wedge \Theta_\alpha & \Omega^F_{\alpha\beta} + |\nabla f|^2 \Theta_\alpha \wedge \Theta_\beta \\ \vdots & \end{array}\right)$$

Tracing over both indices yields

$$\begin{aligned} -\tilde{R} &= \sum_{p,q} \tilde{\Omega}_{pq}(\tilde{E}_p, \tilde{E}_q) \\ &= \sum_{i,j} \tilde{\Omega}_{ij}(\tilde{E}_i, \tilde{E}_j) + \sum_{\alpha,\beta} \tilde{\Omega}_{\alpha\beta}(\tilde{E}_\alpha, \tilde{E}_\beta) + 2\sum_{i,\alpha} \tilde{\Omega}_{i\alpha}(\tilde{E}_i, \tilde{E}_\alpha) \\ &= -R^B - \frac{R^F}{f^2} \left(\frac{|\nabla f|}{f}\right)^2 \underbrace{\sum_{\alpha\neq\beta} (\Theta_\alpha \wedge \Theta_\beta)(E_\alpha, E_\beta)}_{=n(n-1)} + 2\sum_{i,\alpha} (\nabla^2 f(-, E_i) \wedge \Theta_\alpha)(E_i, \frac{E_\alpha}{f}) \\ &= -R^B - \frac{R^F}{f^2} + n(n-1)\left(\frac{|\nabla f|}{f}\right)^2 + 2\sum_\alpha \frac{\Delta f}{f} \\ &= -R^B - \frac{R^F}{f^2} + n(n-1)\left(\frac{|\nabla f|}{f}\right)^2 + \frac{2n\Delta f}{f}. \end{aligned}$$

Multiplying through by -1 gives the result. ■

Exercise 2.3. *(Scalar curvature of Schwarzschild) For $f = f(r)$ and $a > 0$, show the scalar curvature of $\frac{dr^2}{f} + r^2 d\Omega^2_{\mathbb{S}^{n-1}}$ on $[a, \infty) \times \mathbb{S}^{n-1}$ is*

$$\frac{n-1}{r^2}((n-2)(1-f) - rf'),$$

where $d\Omega^2_{\mathbb{S}^{n-1}}$ is the metric on the unit sphere.

Next, we use moving frames to compute the behavior of the scalar curvature under a conformal change.

Proposition 2.4. *Let g be a metric on M^n with $n \geq 3$, then for $u : M \to \mathbb{R}^{>0}$ then $\tilde{g} := u^{\frac{4}{n-2}} g$ has scalar curvature*

$$\tilde{R} = -u^{\frac{n+2}{n-2}} \left(\frac{4(n-1)}{(n-2)} \Delta u - Ru\right).$$

Proof. Take $\{E_i\}$ an orthonormal frame for g, and let $\{\Theta_i\}$ be the induced coframe. Since orthogonality is preserved under a conformal change, $\{\tilde{\Theta}_i\}$ is the induced coframe for $\tilde{g}$, where $\tilde{\Theta}_i = u^{\frac{2}{n-2}}\Theta_i$. We compute the first structure equation as

$$\begin{aligned}
d\tilde{\Theta}_i &= d(u^{\frac{2}{n-2}}\Theta_i) \\
&= d(u^{\frac{2}{n-2}}) \wedge \Theta_i + u^{\frac{2}{n-2}} d\Theta_i \\
&= \frac{2}{n-2} u^{\frac{4-n}{n-2}} du \wedge \Theta_i + \sum_j \omega_{ij} \wedge \tilde{\Theta}_j \\
&= \sum_j \left(-\frac{2}{n-2} u^{-1}(E_j u)\Theta_i + \omega_{ij} \right) \wedge \tilde{\Theta}_j
\end{aligned}$$

As with before,

$$\tilde{\omega}_{ij} \equiv \omega_{ij} - \frac{2}{n-2} u^{-1}(E_j u)\Theta_i \mod \tilde{\Theta}_j.$$

We don't have a second factor to pin down the constant multiple; instead we just need to satisfy anti-symmetry of the connection coefficients. This leads us to anti-symmetrize the second factor as

$$\tilde{\omega}_{ij} = \omega_{ij} - \frac{2}{n-2} u^{-1} \left((E_j u)\Theta_i - (E_i u)\Theta_j \right).$$

We move on to compute the curvature 2-form. Set $f := \frac{2}{n-2}u^{-1}$, so $f' = -\frac{2}{n-2}u^{-2}$ and $f^2 = \left(\frac{2}{n-2}\right)^2 u^{-2}$. Therefore,

$$-\frac{2f'}{n-2} = f^2.$$

We move on to compute the second structure equation as

$$\begin{aligned}
\tilde{\Omega}_{ij} &= d\omega_{ij} \underbrace{- d(f((E_j u)\Theta_i - (E_i u)\Theta_j))}_{\text{(I)}} \\
&\quad - \sum_k \omega_{ik} - f((E_k u)\Theta_i - (E_i u)\Theta_k)) \\
&\qquad \wedge (\omega_{kj} - f((E_j u)\Theta_k - (E_k u)\Theta_j)) \\
&= \underbrace{d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}}_{\text{(C2) on } M} + \text{(I)} \\
&\quad \underbrace{+ f \sum_k \omega_{ik} \wedge ((E_j u)\Theta_k - (E_k u)\Theta_j)}_{\text{(II)}} \\
&\quad \underbrace{+ f \sum_k ((E_k u)\Theta_i - (E_i u)\Theta_k) \wedge \omega_{kj}}_{\text{(III)}} \\
&\quad \underbrace{- f^2 \sum_k ((E_k u)\Theta_i - (E_i u)\Theta_k)) \wedge ((E_j u)\Theta_k - (E_k u)\Theta_j))}_{\text{(IV)}} \\
&= \Omega_{ij} + \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}.
\end{aligned}$$

Since this is a long computation, we break it into parts.

- $$\begin{aligned}
\text{(I)} &= -d(f\,((E_ju)\Theta_i - (E_iu)\Theta_j)) \\
&= -f'du \wedge ((E_ju)\Theta_i - (E_iu)\Theta_j) \\
&\quad - f\Bigg(\sum_k (E_kE_ju)\Theta_k \wedge \Theta_i - (E_kE_iu)\Theta_k \wedge \Theta_j \\
&\quad + (E_ju)d\Theta_i - (E_iu)d\Theta_j\Bigg) \\
&= -f'du \wedge ((E_ju)\Theta_i - (E_iu)\Theta_j) - f\sum_k (E_kE_ju)\Theta_k \wedge \Theta_i \\
&\quad + f\sum_k (E_kE_iu)\Theta_k \wedge \Theta_j - f(E_ju)d\Theta_i + f(E_iu)d\Theta_j. \\
&= \underbrace{\frac{f^2(n-2)}{2}du \wedge ((E_ju)\Theta_i - (E_iu)\Theta_j)}_{(**)} \underbrace{-f\sum_k (E_kE_ju)\Theta_k \wedge \Theta_i}_{(***)} \\
&\quad \underbrace{+f\sum_k (E_kE_iu)\Theta_k \wedge \Theta_j}_{(****)} \underbrace{-f(E_ju)d\Theta_i + f(E_iu)d\Theta_j}_{(*)}.
\end{aligned}$$

- $$\begin{aligned}
\text{(II)} &= f\sum_k \omega_{ik} \wedge ((E_ju)\Theta_k - (E_ku)\Theta_j) \\
&= \underbrace{f(E_ju)d\Theta_i}_{(*)} \underbrace{-f\sum_k (E_ku)\omega_{ik} \wedge \Theta_j}_{(****)}.
\end{aligned}$$

- $$\begin{aligned}
\text{(III)} &= f\sum_k ((E_ku)\Theta_i - (E_iu)\Theta_k) \wedge \omega_{kj} \\
&= f\sum_k \omega_{jk} \wedge ((E_ku)\Theta_i - (E_iu)\Theta_k) \\
&= \underbrace{f\sum_k (E_ku)\omega_{jk} \wedge \Theta_i}_{(***)} \underbrace{-f(E_iu)d\Theta_j}_{(*)}.
\end{aligned}$$

- $$\begin{aligned}
\text{(IV)} &= -f^2\sum_k ((E_ku)\Theta_i - (E_iu)\Theta_k)) \wedge ((E_ju)\Theta_k - (E_ku)\Theta_j)) \\
&= -f^2\sum_k ((E_ku)(E_ju)\Theta_i \wedge \Theta_k - (E_ku)^2\Theta_i \wedge \Theta_j + (E_iu)(E_ku)\Theta_k \wedge \Theta_j \\
&= f^2(E_ju)\sum_k (E_ku)\Theta_k \wedge \Theta_i - f^2(E_iu)\sum_k (E_ku)\Theta_k \wedge \Theta_j + f^2|\nabla u|^2\Theta_i \wedge \Theta_j \\
&= f^2(E_ju)du \wedge \Theta_i - f^2(E_iu)du \wedge \Theta_j + f^2|\nabla u|^2\Theta_i \wedge \Theta_j \\
&= \underbrace{f^2du \wedge ((E_ju)\Theta_i - (E_iu)\Theta_j)}_{(**)} + f^2|\nabla u|^2\Theta_i \wedge \Theta_j
\end{aligned}$$

Recombining (I)-(IV), we see that [Proposition 3.1]

1. $(*) = 0$,
2. $(**) = \frac{nf^2}{2} du \wedge ((E_j u)\Theta_i - (E_i u)\Theta_j)$,
3. $(***) = -f\nabla^2 u(-, E_j) \wedge \Theta_i$,
4. $(****) = f\nabla^2 u(-, E_i) \wedge \Theta_j$.

In total, the new curvature 2-form is

$$\tilde{\Omega}_{ij} = \Omega_{ij} + \underbrace{f(\nabla^2 u(-, E_i) \wedge \Theta_j - \nabla^2 u(-, E_j) \wedge \Theta_i)}_{\text{(i)}}$$
$$+ \underbrace{\frac{nf^2}{2} du \wedge ((E_j u)\Theta_i - (E_i u)\Theta_j)}_{\text{(ii)}} + \underbrace{f^2 |\nabla u|^2 \Theta_i \wedge \Theta_j}_{\text{(iii)}}.$$

We now trace over i, j to compute the negative of the scalar curvature. Again, we compute each part separately. Recalling that $\tilde{E}_i = u^{\frac{-2}{n-2}} E_i$, we compute

1. $$\begin{aligned} \mathrm{Tr(i)} &= u^{\frac{-4}{n-2}} f \sum_{i,j} (\nabla^2 u(-, E_i) \wedge \Theta_j - \nabla^2 u(-, E_j) \wedge \Theta_i)(E_i, E_j) \\ &= u^{\frac{-4}{n-2}} \left(\frac{2}{n-2} u^{-1} \right) 2 \sum_{i \neq j} \nabla^2 u(E_i, E_i) \cdot \Theta_j(E_j) \\ &= \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \Delta u, \end{aligned}$$
2. $$\begin{aligned} \mathrm{Tr(ii)} &= u^{\frac{-4}{n-2}} \frac{nf^2}{2} \sum_{i,j} du \wedge ((E_j u)\Theta_i - (E_i u)\Theta_j)(E_i, E_j) \\ &= u^{-\frac{4}{n-2}} nf^2 \sum_{i \neq j} -(E_j u)(\Theta_i \wedge du)(E_i, E_j) \\ &= u^{-\frac{4}{n-2}} nf^2 \sum_{i \neq j} -(E_j u)^2 \Theta_i(E_i) \\ &= -u^{-\frac{4}{n-2}} f^2 |\nabla u|^2 n(n-1), \end{aligned}$$
3. $$\begin{aligned} \mathrm{Tr(iii)} &= u^{\frac{-4}{n-2}} f^2 |\nabla u|^2 \sum_{i,j} (\Theta_i \wedge \Theta_j)(E_i, E_j) \\ &= u^{\frac{-4}{n-2}} f^2 |\nabla u|^2 n(n-1). \end{aligned}$$

Since Tr(ii) + Tr(iii) = 0, we see that

$$-\tilde{R} = -Ru^{\frac{-4}{n-2}} + \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \Delta u.$$

The result follows after some basic algebraic manipulations. ■

3 Derivations

Proposition 3.1. *Let $\{E_i\}$ denote an orthonormal frame and $\{\Theta_i\}$ its dual coframe. Let f denote a function.*

1. $\nabla f = \sum_i (E_i f) E_i$.
2. $\nabla^2 f(-, E_i) = \sum_j (E_j E_i f)\Theta_j - (E_j f)\omega_{ij}$.

Proof. 1. $\langle \nabla f, E_k \rangle = E_k f \implies \nabla f = \sum_k (E_k f) E_k$.

2. $$\begin{aligned}\nabla^2 f(E_i, E_j) &:= \langle \nabla_{E_i} \nabla f, E_j \rangle \\ &= \left\langle \nabla_{E_i} \sum_k (E_k f) E_k, E_j \right\rangle \\ &= \sum_k \langle (E_i E_k f) E_k + (E_k f) \nabla_{E_i} E_k, E_j \rangle \\ &= E_i E_j f + \sum_{k,l} \langle (E_k f) \omega_{kl}(E_i) E_l, E_j \rangle \\ &= E_i E_j f + \sum_k (E_k f) \omega_{kj}(E_i) \\ &= \sum_k (E_k E_j f) \Theta_k(E_i) + (E_k f) \omega_{kj}(E_i).\end{aligned}$$

Therefore, the Hessian with one slot filled is the following $(0,1)$-tensor

$$\nabla^2 f(-, E_j) = \sum_k (E_k E_j f)\Theta_k - (E_k f)\omega_{jk}.$$

Re-indexing gives the desired conclusion. As a bonus, we express the $(0,2)$ Hessian as

$$\nabla^2 f = \sum_{k,l} (E_k E_l f)\Theta_k \otimes \Theta_l - (E_k f)\omega_{lk} \otimes \Theta_l.$$

■

Proposition 3.2. *(Cartan's 1^{st} Structure Equation) Given an orthonormal coframe $\{\Theta_i\}$,*

$$d\Theta_i(X, Y) = \sum_j \omega_{ij} \wedge \Theta_j.$$

Proof. We begin by computing

$$\nabla_X Y = \nabla_X (\sum_j \Theta_j(Y) E_j) = \sum_j X(\Theta_j(Y)) E_k + \sum_{j,k} \Theta_j(Y) \omega_{jk}(X) E_k.$$

Applying Θ_i to the equality gives

$$\Theta_i(\nabla_X Y) = X(\Theta_i(Y)) + \sum_j \Theta_j(Y) \omega_{ji}(X).$$

Now,

$$\begin{aligned}
d\Theta_i(X,Y) &= X(\Theta_i(Y)) - Y(\Theta_i(X)) - \Theta_i([X,Y]) \\
&= \Theta_i(\nabla_X Y) - \sum_j \Theta_j(Y)\omega_{ji}(X) - \Theta_i(\nabla_Y X) + \sum_j \Theta_j(X)\omega_{ji}(Y) - \Theta_i([X,Y]) \\
&= \underbrace{\tau^i(X,Y)}_{=0} + \sum_j \omega_{ij}(X)\Theta_j(Y) - \omega_{ij}(Y)\Theta_j(X) \\
&= \sum_j (\omega_{ij} \wedge \Theta_j)(X,Y).
\end{aligned}$$

■

Proposition 3.3. *(Cartan's 2^{nd} Structure Equation) For X, Y vector fields, $\{E_i\}$ an orthonormal frame, and R the (1,3) curvature tensor, we have*

$$\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Proof.

$$\begin{aligned}
\sum_j \Omega_{ij}(X,Y)E_j &= R(X,Y)E_i \\
&:= \nabla_X\nabla_Y E_i - \nabla_Y\nabla_X E_i - \nabla_{[X,Y]}E_i \\
&= \sum_j \nabla_X(\omega_{ij}(Y)E_j) - \nabla_Y(\omega_{ij}(X)E_j) - \omega_{ij}([X,Y])E_j \\
&= \sum_{j,k} X(\omega_{ij}(Y))E_j + \omega_{ij}(Y)\omega_{jk}(X)E_k - Y(\omega_{ij}(X))E_j \\
&\qquad - \omega_{ij}(X)\omega_{jk}(Y)E_k - \omega_{ij}([X,Y])E_j \\
&= \sum_j d\omega_{ij}(X,Y)E_j + \sum_k \left(\sum_j \omega_{jk}(X)\omega_{ij}(Y) - \omega_{ij}(X)\omega_{jk}(Y) \right) E_k \\
&= \sum_j d\omega_{ij}(X,Y)E_j - \sum_k \sum_j (\omega_{ij} \wedge \omega_{jk})(X,Y)E_k. \\
&= \sum_j d\omega_{ij}(X,Y)E_j - \sum_k (\omega_{ik} \wedge \omega_{kj})(X,Y)E_j.
\end{aligned}$$

Following through with Θ_j gives the desired conclusion. ■

Proposition 3.4. *Both ω_{ij} and Ω_{ij} are anti-symmetric in its indices.*

Proof. By (C2), it is clear that Ω_{ij} is anti-symmetric in its indices if ω_{ij} is. Alternatively, one can argue using straight from the definition of Ω_{ij} via the anti-symmetry of the Riemann curvature tensor. To show symmetry of the connection 1-forms, for any vector field X,

$$\nabla_X E_i = \sum_j \omega_{ij}(X)E_j \implies \langle \nabla_X E_i, E_j \rangle = \omega_{ij}(X).$$

By metric compatibility,

$$0 = X\langle E_i, E_j\rangle = \langle \nabla_X E_i, E_j\rangle + \langle E_i, \nabla_X E_j\rangle = \omega_{ij}(X) + \omega_{ji}(X).$$

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P. Tee, Department of Mathematics, University of Connecticut, Connecticut 06269

E-mail address, P. Tee: paul.tee@uconn.edu